Weyl-Type Theorems for Unbounded Operators

Anuradha Gupta

Department of Mathematics,

Delhi College of Arts and Commerce,

University of Delhi.

1. Introduction

In 1909, H. Weyl (*Über beschränkte quadatische Formen, deren Differenz Vollstetig ist*) examined the spectra of all compact perturbations of a self adjoint operator on a Hilbert space and found that their intersection consisted precisely of those points of the spectrum which were not isolated eigenvalues of finite multiplicity. A bounded linear operator satisfying this property is said to satisfy **Weyl's Theorem**.

Further, in 2002, M. Berkani (*Index of B-Fredholm operators and generalization of a Weyl's theorem*) proved that if T is a bounded normal operator acting on a Hilbert space H then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, where E(T) is the set of all isolated eigenvalues of T, which gives the generalization of the Weyl's Theorem. He also proved this generalized version of classical Weyl's Theorem for bounded hyponormal operators (*Generalized Weyl's theorem and hyponormal operators*, 2004).

Following Weyl and Berkani, various variants of Weyl's Theorem, generally known as the Weyl-type theorems, have been introduced with much attention to an approximate point version called a-Weyl's theorem.

Study of other generalizations began in 2003 that resulted in various Weyl type theorems, viz., Browder's theorem, a-Browder's theorem, generalized Weyl's theorem, generalized a-Weyl's theorem, generalized Browder's theorem and generalized a-Browder's theorem.

This study, however, was limited to the classes of bounded operators.

2. Classes of Linear operators

Bounded Linear operators

Let H be a complex Hilbert space and $\mathrm{T} : \mathcal{D}(\mathrm{T}) \longrightarrow H$, a linear transformation from a linear subspace $\mathcal{D}(\mathrm{T})$ of H into H. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathcal{D}(\mathrm{T})$, $\|\mathrm{T}x\| \leqslant \mathrm{c}\|x\|$.

Bounded Normal operators

A **Normal operator** on a complex Hilbert Space H is a continuos linear operators $\mathrm{N} : H \to H$ that commutes with its adjoint operators N^*, that is, $\mathrm{NN}^* = \mathrm{N}^*\mathrm{N}$.

Several attempts have been made to generalize the classes of normal operators by weakening the commutativity requirement.

Non-Normal classes of operators

- Quasi-normal operators: $N(N^*N) = (N^*N)N$.
- Hyponormal operators: $N^*N \geqslant NN^*$ or equivalently, $\|Nx\| \geqslant \|N^*x\|$ for every unit vector $x \in H$.
- Paranormal operators: $\|N^2x\| \geqslant \|Nx\|^2$ for every unit vector $x \in H$.
- *-Paranormal operators: $\|N^2x\|.\|x\| \geqslant \|N^*x\|^2$ for every $x \in H$.

It is well known that the following containments hold:

Normal $\subset$ Quasi-normal $\subset$ Hyponormal $\subset$ Paranormal $\subset$ *-Paranormal

Unbounded linear operators

Let H be a complex Hilbert space. An operator $\mathrm{T} : \mathcal{D}(\mathrm{T}) \longrightarrow H$ is said to be **unbounded** if for every c there exists an x in $\mathcal{D}(\mathrm{T})$ such that $\|\mathrm{T}x\| \geqslant \mathrm{c}\|x\|$.

Densely defined operators

An operator T is said to be **densely defined** if its domain is dense in H.

The denseness of the domain is necessary and sufficient for the existence of the adjoint and the transpose.

Most applications use unbounded linear operators which are closed or atleast have closed linear extensions.

Closed linear operators

An operator $\mathrm{T} : \mathcal{D}(\mathrm{T}) \longrightarrow H$ is said to be a **closed linear operator** if $x_n \longrightarrow x$ $(x_n \in \mathcal{D}(\mathrm{T}))$ and $\mathrm{T}x_n \longrightarrow y$ together imply that $x \in \mathcal{D}(\mathrm{T})$ and $\mathrm{T}x = y$. We denote the class of all closed linear operators by $C(H)$.

Note: If an operator T is closed, densely defined and continuous on its domain, then it is defined on all of H.

3. Spectral theory

If T is a bounded linear operator, it is well known that the spectrum of T, $\sigma(T)$ is the set of all those $\lambda \in \mathbb{C}$ for which T - λI is not bijective. This concept can be extended to the case when T is a densely defined unbounded linear operator.

Spectrum of an unbounded linear operator

Let T be a densely defined unbounded linear operator. A complex number λ is said to be in the spectrum of T if T - λI does not have a bounded inverse.

Remark

The spectrum of an unbounded operator is in general a closed (possibly empty) subset of the complex plane unlike the case of bounded operators where the spectrum is a non-empty, closed and bounded subset of $\mathbb{C}$.

Remark

Boundedness of the inverse operator follows directly from its existence in case if T is a closed linear operator. This follows from the closed graph theorem. Therefore, as in the case of bounded operators, spectrum of a closed linear operator is also the set of those λ for which T - λI is not bijective.

The ascent and Descent of an operator

The **ascent** p(T) and **descent** q(T) of an operator $T \in C(H)$ are given by

$$p(T) = \inf\{n : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\} \text{ and}$$
$$q(T) = \inf\{n : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}.$$

It is well known that the resolvent operator $R_\lambda(T) = (T - \lambda I)^{-1}$ is an analytic function for all $\lambda \in \rho(T)$ and the isolated points of $\sigma(T)$ are either poles or essential singularities of $R_\lambda(T)$.

Pole of the resolvent operator

For $T \in C(H)$ an isolated point $\lambda \in \sigma(T)$ is said to be a **pole of order p** if $p = p(T - \lambda I) < \infty$ and $q(T - \lambda I) < \infty$.

Left-Pole of the resolvent operator

A point $\lambda \in \sigma_a(T)$ is said to be a *left-pole* if $p = p(T - \lambda I) < \infty$ and $\mathcal{R}(T - \lambda I)^{p+1}$ is closed.

If $\pi(T)$ & $\pi^a(T)$ denote the set of all poles and left-poles of T, respectively, and $\pi_o(T)$ & $\pi_o^a(T)$ denote the set of all poles and left-poles of finite multiplicity, respectively, then clearly:

$$\pi(T) \subseteq \pi^a(T) \text{ and } \pi_o(T) \subseteq \pi_o^a(T).$$

We have the following necessary and sufficient condition due to Lay (*Spectral Analysis Using Ascent, Descent, Nullity and Defect*, 1970):

Let T be a closed linear operator with $\rho(\mathrm{T}) \neq \emptyset$. If $\lambda_o \in \sigma(\mathrm{T})$ and there exists two closed subspaces M and N such that T - λ_oI is one-one mapping of $\mathcal{D}(\mathrm{T}) \cap M$ onto M, T - $\lambda_o \mathrm{I}|_N$ is nilpotent of index p and $H = M \oplus N$, then $M = \mathcal{R}(\mathrm{T} - \lambda_o \mathrm{I})^p$, $N = \mathcal{N}(\mathrm{T} - \lambda_o \mathrm{I})^p$ and λ_o is a pole of the resolvent $\mathrm{R}_\lambda(\mathrm{T})$ of order p. The above condition is also necessary.

4. Weyl theory

Semi-Fredholm operators

Let T $\in C(H)$ and let $\mathcal{R}$(T) and $\mathcal{N}$(T) denote the range and null space of T, respectively. If $\mathcal{R}$(T) is closed and nullity of T, α(T) = dim $\mathcal{N}$(T) $< \infty$ (resp., defect of T, β(T) = codim $\mathcal{R}$(T) $< \infty$) then T is called an **upper semi-Fredholm** (resp. **lower semi-Fredholm**) operator. A **semi-Fredholm operator** is an upper or lower semi-Fredholm operator.

By $SF_+(H)$ (resp. $SF_-(H)$) we denote the class of upper (resp. lower) semi-Fredholm operators. For T $\in SF_+(H) \cup SF_-(H)$, **index of T** is defined as $ind(\text{T}) = \alpha(\text{T})$ - $\beta(\text{T})$.

We have,

$SF_+^-(H) = \{\text{T} \in C(H) : \text{T} \in SF_+(H)$ and $ind(\text{T}) \leqslant 0 \}$, and
$SF_-^+(H) = \{\text{T} \in C(H) : \text{T} \in SF_-(H)$ and $ind(\text{T}) \geqslant 0 \}$

and these operators generate the following spectrum

$\sigma_{SF_+^-}(\text{T}) = \{\lambda \in \mathbb{C} : \text{T - }\lambda\text{I} \notin SF_+^-(H) \}$ and
$\sigma_{SF_-^+}(\text{T}) = \{\lambda \in \mathbb{C} : \text{T - }\lambda\text{I} \notin SF_-^+(H) \}$

Fredholm operators

If both α(T) and β(T) are finite then T is called a **Fredholm operator**. An operator $T \in C(H)$ is called **Weyl** if it is Fredholm of index 0 and the **Weyl spectrum** of T is defined as $\sigma_W(\mathrm{T}) = \{\lambda \in \mathbb{C} :$ T - λI is not Weyl$\}$.

The concept of Fredholm operators was generalized by Berkani to the class of unbounded B-Fredholm operators (*Unbounded B-Fredholm operators on Hilbert spaces*, 2008).

Let $\mathrm{T} \in C(H)$ and let $\Delta(\mathrm{T}) = \{n \in \mathbb{N} \colon \forall m \in \mathbb{N},\ m \geqslant n \Rightarrow \mathcal{R}(\mathrm{T}^n) \cap \mathcal{N}(\mathrm{T}) \subseteq \mathcal{R}(\mathrm{T}^m) \cap \mathcal{N}(\mathrm{T})\}$. Then the **degree of stable iteration** of T is defined as $\mathrm{dis}(\mathrm{T}) = \inf \Delta(\mathrm{T})$ where $\mathrm{dis}(\mathrm{T}) = \infty$ if $\Delta(\mathrm{T}) = \emptyset$.

Semi B-Fredholm operators

Let $\mathrm{T} \in C(H)$ be densely defined on H. We say that $\mathrm{T} \in C(H)$ is **semi B-Fredholm operator** if it is either upper or lower semi B-Fredholm operator, where T is an upper (resp. lower) semi B-Fredholm operator if there exists an integer $d \in \Delta(\mathrm{T})$ such that $\mathcal{R}(\mathrm{T}^d)$ is closed and $\dim\{\mathcal{N}(\mathrm{T}) \cap \mathcal{R}(\mathrm{T}^d)\} < \infty$ (resp. $\operatorname{codim}\{\mathcal{R}(\mathrm{T}) + \mathcal{N}(\mathrm{T}^d)\} < \infty$).

In either case, **index of T** is defined as the number

$$ind(\mathrm{T}) = \dim \{\mathcal{N}(\mathrm{T}) \cap \mathcal{R}(\mathrm{T}^d)\} \ - \ \operatorname{codim} \{\mathcal{R}(\mathrm{T}) + \mathcal{N}(\mathrm{T}^d)\}.$$

Let $SBF_+(H)$ denote the class of all upper semi B-Fredholm operators. Then $SBF_+^-(H) = \{\mathrm{T} \in C(H) : \mathrm{T} \in SBF_+(H)$ and $\mathrm{ind}(\mathrm{T}) \leqslant 0\}$ and $\sigma_{SBF_+^-}(H) = \{\lambda \in \mathbb{C} : \mathrm{T} - \lambda \mathrm{I} \notin SBF_+^-(H)\}$.

B-Fredholm operators:

We say that T is **B-Fredholm operator** if T is both upper and lower semi B-Fredholm operator, that is, there exists an integer $d \in \Delta(T)$ such that T satisfies the following conditions:

(i) $\dim \{\mathcal{N}(T) \cap \mathcal{R}(T^d)\} < \infty$

(ii) $\operatorname{codim} \{\mathcal{R}(T) + \mathcal{N}(T^d)\} < \infty$

B-Weyl spectrum

An operator T $\in C(H)$ is said to be **B-Weyl** if it is a B-Fredholm operator of index zero and the **B-Weyl spectrum** of T is defined as $\sigma_{BW}(\mathrm{T}) = \{\lambda \in \mathbb{C} : \mathrm{T} - \lambda \mathrm{I}$ is not B-Weyl$\}$.

Browder operators

An operator T $\in C(H)$ is said to be **upper semi-Browder** (resp. **lower semi-Browder**) if T is upper semi-Fredholm with p(T) $< \infty$ (resp. lower semi-Fredholm with q(T) $< \infty$). If T is both upper and lower semi-Browder, that is, if T is a Fredholm operator with ascent and descent both finite, then T is **Browder**.

Browder spectrum:

The upper-Browder, lower-Browder and Browder spectra are defined as

$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not upper semi-Browder}\}$,

$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not lower semi-Browder}\}$ and

$\sigma_{b}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ not Browder}\}$, respectively.

Clearly, $\sigma_{SF_+^-}(T) \subseteq \sigma_{ub}(T)$ and $\sigma_w(T) \subseteq \sigma_b(T)$.

One of the interesting properties in Fredholm theory is the single valued extension property (SVEP). This property was first introduced by Dunford (1952). Mainly we concern with the localized version of SVEP, the SVEP at a point, introduced by Finch (1975) and relate it to the finiteness of the ascent of an operator.

The Single valued extension property

Let T : $\mathcal{D}$(T) $\subset H \rightarrow H$ be a closed linear mapping and let λ_o be a complex number. The operator T has the *single valued extension property* (SVEP) at λ_o if $f = 0$ is the only solution to (T- λI)$f(\lambda) = 0$ that is analytic in a neighborhood of λ_o. Also, T has SVEP if it has this property at every point λ_o in the complex plane.

We have the following relation between the SVEP at a point and the ascent of an operator:

Let T $\in C(H)$.

(i) If p(T - λI) is finite for some $\lambda \in \mathbb{C}$, then T has SVEP at λ.

(ii) If T is onto and not one-one, then T does not have SVEP at $\lambda = 0$.

The second condition can also be rephrased as "If T has SVEP, then T is invertible whenever it is onto, that is, $\sigma(T) = \sigma_s(T)$".

5. Weyl Type Theorems for Unbounded Operators

By isoσ(T) and isoσ_a(T) we denote the isolated points of σ(T) and σ_a(T), respectively. We use the following notations:

$E(T)$: the set of all eigenvalues in iso$\sigma(T)$,

$E_o(T)$: the set of all eigenvalues of finite multiplicities in iso$\sigma(T)$,

$E^a(T)$: the set of all eigenvalues in iso$\sigma_a(T)$,

$E_o^a(T)$: the set of all eigenvalues of finite multiplicities in iso$\sigma_a(T)$.

Following are some of the variants of Weyl's Theorem.

Weyl-type theorems

We say that $T \in C(H)$ satisfies:

(i) Weyl's Theorem if $\sigma(T) \setminus \sigma_w(T) = E_o(T)$.

(ii) Generalized Weyl's Theorem if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$.

(iii) a-Weyl's Theorem if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_o^a(T)$.

(iv) Generalized a-Weyl's Theorem if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$.

(v) Browder's Theorem if $\sigma(T) \setminus \sigma_w(T) = \pi_o(T)$.

(vi) Generalized Browder's Theorem if $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$.

(vii) a-Browder's Theorem if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_o^a(T)$.

(viii) Generalized a-Browder's Theorem

Weyl-type theorems

(ix) property (w) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_o(T)$.

(x) property (gw) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$.

(xi) property (aw) if $\sigma(T) \setminus \sigma_w(T) = E_o^a(T)$.

(xii) property (gaw) if $\sigma(T) \setminus \sigma_{BW}(T) = E^a(T)$.

(xiii) property (b) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_o(T)$.

(xiv) property (gb) if $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi(T)$.

(xv) property (ab) if $\sigma(T) \setminus \sigma_w(T) = \pi_o^a(T)$.

(xvi) property (gab) if $\sigma(T) \setminus \sigma_{BW}(T) = \pi^a(T)$.

5.1 Weyl Type Theorems for Unbounded Normal Operators

Theorem:

If T is a closed normal operator, then:

(i) $\|Tx\| = \|T^*x\|, \ \forall \ x \in \mathcal{D}(T) = \mathcal{D}(T^*)$

(ii) $H = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T)$

(iii) $p(T) = 0$ or 1.

Remark:

If T is normal then so are T^* (the adjoint of T), $T - \lambda I$ and $T^* - \lambda I$ for all $\lambda \in \mathbb{C}$. Therefore, whenever T is a closed normal operator, (i) (ii) and (iii) of the above theorem hold with T replaced by T^*, $T - \lambda I$ and $T^* - \lambda I$ for all $\lambda \in \mathbb{C}$.

Result:

Let T be a closed normal operator with $\mathcal{D}(T) \subset H$. Then $\sigma(T) = \sigma_a(T)$

let $\varrho(H) = \{T \in C(H) :$ T is a densely defined unbounded normal operator with $\rho(T) \neq \phi\}$

Theorem:

If $T \in \varrho(H)$, then λ is an isolated point of $\sigma(T)$ iff λ is a simple pole of the resolvent of T.

Corollary:

If $T \in \varrho(H)$, then $\pi(T) = E(T) = E^a(T)$ and in particular, $\pi_o(T) = E_o(T) = E^a_o(T)$.

Theorem:

Let $T \in \varrho(H)$. Then the following are equivalent:

(i) a-Weyl's Theorem

(ii) Weyl's Theorem

(iii) Browder's Theorem

(iv) a-Browder's Theorem

Theorem:

Let $T \in \varrho(H)$. Then the following are equivalent:

(i) Generalized a-Weyl's Theorem

(ii) Generalized Weyl's Theorem

(iii) Generalized Browder's Theorem

(iv) Generalized a-Browder's Theorem

Theorem:

Let $T \in \varrho(H)$. Then the following are equivalent:

(i) property (aw)

(ii) property (w)

(iii) property (b)

(iv) property (ab)

Theorem:

Let $T \in \varrho(H)$. Then the following are equivalent:

(i) property (gaw)

(ii) property (gw)

(iii) property (gb)

(iv) property (gab)

Example

Let $H = l^2$ and let T be defined as follows:

$$T(x_1, x_2, x_3, \ldots) = (ix_1, 2x_2, x_3, 4x_4, x_5, \ldots) \qquad (i = \sqrt{-1})$$

$$= (a_1x_1, a_2x_2, a_3x_3, a_4x_4, a_5x_5, \ldots),\ say,$$

$$where, \quad a_j = \begin{cases} i, & j = 1; \\ j, & \text{if } j = 2n,\ n \in \mathbb{N}; \\ 1, & \text{if } j = 2n+1,\ n \in \mathbb{N}. \end{cases}$$

$$\text{and } \mathcal{D}(T) = \left\{ (x_1, x_2, x_3, \ldots) \in l^2 : \sum_{j=1}^{\infty} |a_j x_j|^2 < \infty \right\}.$$

Then T is a closed, densely defined, unbounded normal operator with $\sigma(T) = \sigma_a(T) = \sigma_p(T) = \{a_j : j \in \mathbb{N}\} = \{i, 1, 2, 4, 6, \dots\}$. Also,

$$E_o(T) = \{i, 2, 4, 6, \dots\} = E_o^a(T) = \pi_o(T) = \pi_o^a(T),$$

$$E(T) = \{i, 1, 2, 4, 6, \dots\} = E^a(T) = \pi(T) = \pi^a(T),$$

$$\sigma_w(T) = \{1\} = \sigma_{SF_+^-}(T),$$

$$\sigma_{BW}(T) = \phi = \sigma_{SBF_+^-}(T).$$

5.2 Weyl Type Theorems for Unbounded Hyponormal Operators

Lemma:

Let T $\in C(H)$ be an unbounded hyponormal operator. Then:

(i) the ascent p(T - λI) $=$ 0 or 1, for every $\lambda \in \mathbb{C}$

(ii) λ is an isolated point of $\sigma(T)$ iff λ is a pole of the resolvent of T.

NOTE: (ii) can also be rephrased as " Every unbounded Hyponormal operator is a **polaroid** operator."

Theorem:

Let T be an unbounded Hyponormal operator. Then:

(i) T satisfies Weyl's Theorem

(ii) T satisfies Browder's Theorem

(iii) T satisfies generalized Weyl's Theorem

(iv) T and T^* satisfy a-Browder's Theorem.

Example

Let $H = l^2$ and let T be defined as:

$$T(x_1, x_2, x_3, \dots) = (0, x_1, 2x_2, 3x_3, 4x_4, \dots)$$
$$= (0, a_1x_1, a_2x_2, a_3x_3, a_4x_4, \dots)$$

where, $a_n = n$ for all $n \in \mathbb{N}$ and

$$\mathcal{D}(\mathrm{T}) = \left\{(x_1, x_2, x_3, \dots) \in l^2 : \sum_{j=1}^{\infty} |a_j x_j|^2 < \infty\right\}.$$

Then T is an unbounded hyponormal operator.

We have, $\sigma_p(T) = \phi$. Therefore, $E(T) = E_o(T) = \phi$. Also, $\sigma(T) = \mathbb{C} \cup \{\infty\}$, $\sigma_a(T) = \{\infty\}$.

$\sigma_w(T) = \mathbb{C} \cup \{\infty\} = \sigma_b(T)$ and hence T satisfies Browder's Theorem,

$\sigma_{SF_+^-}(T) = \{\infty\} = \sigma_{ub}(T)$ and hence T satisfies a-Browder's Theorem,

$\sigma(T) \setminus \sigma_w(T) = \phi = E_o(T)$ and hence T satisfies Weyl's Theorem, and

$\sigma(T) \setminus \sigma_{BW}(T) = \phi = E(T)$ and hence T satisfies generalized Weyl's Theorem.

Let $\Re(H) = \{\mathrm{T} \in C(H)$: T is an unbounded hyponormal operator with the resolvent set $\rho(\mathrm{T}) \neq \emptyset\}$.

It is known that property (gw) implies property (gb) and property (w) implies property (b) for every $\mathrm{T} \in B(H)$, but the converse of these results do not hold true in general (Berkani & Zariouh, 2009). However, we have proved the following equivalence:

Theorem:

Let $\mathrm{T} \in \Re(H)$. Then property (b) holds for T iff property (w) holds for T.

Also, it is shown that for every $\mathrm{T} \in \Re(H)$, $\mathrm{E}(\mathrm{T}) = \pi(\mathrm{T})$ and in particular, $\mathrm{E}_o(\mathrm{T}) = \pi_o(\mathrm{T})$. This helped establish the following equivalences:

Theorem

Let $\mathrm{T} \in \Re(H)$. Then:

(i) generalized Weyl's Theorem is equivalent to generalized Browder's Theorem

(ii) Weyl's Theorem is equivalent to Browder's Theorem

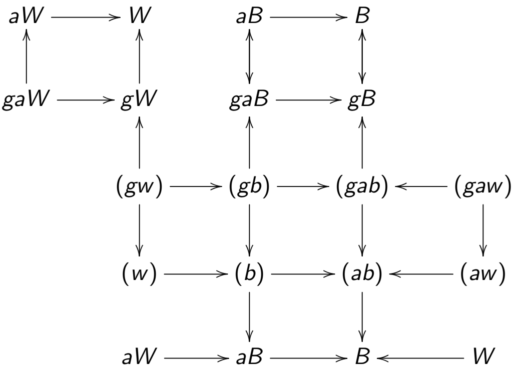
(iii) property (gw) is equivalent to property (gb).

Remark: If $\mathrm{T} \in \Re(H)$, then T satisfies generalized Browder's Theorem.

6. Summary

To summarize, we use the abbreviations gaW, aW, gW, W, and (gaw), (aw), (gw), (w) to signify that an operator $T \in \varrho(H)$ satisfies generalized a-Weyl's Theorem, a-Weyl's Theorem, generalized Weyl's Theorem, Weyl's Theorem, respectively and properties (gaw), (aw), (gw), (w) respectively. Similarly abbreviations gaB, aB, gB, B, and (gab), (gb), (gb), (b) have analogous meaning with respect to Browder-type theorems and properties.

The following diagram shows the relations between several Weyl-type theorems, Browder-type theorems and properties for a **bounded linear operator T**. The arrows signify implications between the theorems and properties.



The following diagram shows the relations that hold between various variants of Weyl's Theorem when $\mathbf{T} \in \varrho(\mathbf{H})$.

$$
\begin{array}{ccccccccc}
aW & \Longleftrightarrow & W & \Longleftrightarrow & aB & \Longleftrightarrow & B & & \\
\Big\uparrow & & \Big\uparrow & & \Big\updownarrow & & \Big\updownarrow & & \\
gaW & \Longleftrightarrow & gW & \Longleftrightarrow & gaB & \Longleftrightarrow & gB & & \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
 & & (gw) & \Longleftrightarrow & (gb) & \Longleftrightarrow & (gab) & \Longleftrightarrow & (gaw) \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
 & & (w) & \Longleftrightarrow & (b) & \Longleftrightarrow & (ab) & \Longleftrightarrow & (aw)
\end{array}
$$

In this diagram, we consider $\mathbf{T} \in \Re(\mathbf{H})$ and we notice that several one sided implications now become equivalences.

$$
\begin{array}{ccccccccc}
gW & \longleftarrow & (gw) & \Longleftrightarrow & (gb) & \longrightarrow & gB & \Longleftrightarrow & gW \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (w) & \Longleftrightarrow & (b) & \longrightarrow & B & \Longleftrightarrow & W \\
 & & \downarrow & & & & \uparrow & & \\
 & & W & & & & aB & &
\end{array}
$$

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